

VECTOR BUNDLES OF RANK FOUR AND $A_3 = D_3$

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ABSTRACT. Over a scheme X with 2 invertible, we show that a version of “Pascal’s rule” for vector bundles of rank 4 gives an explicit isomorphism between the moduli functors represented by projective homogeneous bundles for reductive group schemes of type A_3 and D_3 . We exploit this to prove that a locally free \mathcal{O}_X -module \mathcal{V} of rank 4 has a sub or quotient invertible sheaf if and only if the canonical symmetric bilinear form on $\bigwedge^2 \mathcal{V}$ has a lagrangian subspace. Under additional hypotheses on X (e.g. proper over a field), we prove that this is equivalent to the vanishing of the Witt-theoretic Euler class $e(\mathcal{V}) \in W^4(X, \det \mathcal{V}^\vee)$.

INTRODUCTION

Let X be a noetherian scheme with 2 invertible and \mathcal{V} a locally free \mathcal{O}_X -module of even rank $r = 2s$. The middle exterior power $\bigwedge^s \mathcal{V}$ supports a canonical regular $\det(\mathcal{V})$ -valued $(-1)^s$ -symmetric bilinear form

$$\bigwedge^s \mathcal{V} \otimes \bigwedge^s \mathcal{V} \xrightarrow{\wedge} \det \mathcal{V}$$

given by wedging. This operation defines a functor from the category of locally free \mathcal{O}_X -modules of even rank to the category of regular line bundle-valued bilinear forms and can easily be generalized to categories of perfect complexes. When \mathcal{V} is free, the associated middle exterior form is hyperbolic. On the other hand, for example when $X = \mathbb{P}^4$ and $\mathcal{V} = \Omega_{\mathbb{P}^4}^1$ (see Walter [27]), the middle exterior power form need not even be metabolic. In this work, we give a general necessary condition for the middle exterior form to be metabolic, namely, that \mathcal{V} has a sub or quotient invertible sheaf. This is related to a version of “Pascal’s rule” (see Lemma 1.4) for vector bundles. When \mathcal{V} has rank 4, we prove (see Corollary 3.4) that this condition is sufficient as well.

Theorem 1. *Let X be a noetherian scheme with 2 invertible and \mathcal{V} a locally free \mathcal{O}_X -module of rank 4. Then \mathcal{V} has a sub or quotient invertible sheaf if and only if the middle exterior form on $\bigwedge^2 \mathcal{V}$ is metabolic.*

This involves the (quite classical) geometry of the symmetric lagrangian grassmannian and the exceptional isomorphism $A_3 = D_3$. We then give an interpretation of this result involving the Euler class in Witt and Grothendieck–Witt theory (in the sense of Fasel–Srinivas [18]). What follows is a special case of Corollary 3.8.

Theorem 2. *Let X be a noetherian scheme proper over a complete local ring with residue field of characteristic $\neq 2$. Let \mathcal{V} be a locally free \mathcal{O}_X -module of rank 4. Then \mathcal{V} has a sub or quotient invertible sheaf if and only if the Witt-theoretic Euler class $e(\mathcal{V}) \in W^4(X, \det \mathcal{V}^\vee)$ vanishes.*

This follows from a combination of Theorem 1, explicit formulas for Euler classes, and general results about the Witt cancellation property in exact categories with duality (found in Knus [23, II §6]).

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With no assumption on 2, there is a natural *quadratic* form $\bigwedge^s \mathcal{V} \rightarrow \det \mathcal{V}$, see [24, II Prop. 10.12]. Likewise, replacing \mathcal{V} by an Azumaya algebra \mathcal{A} of degree $r = 2s$, there is the notion of the *canonical involution* on $\lambda^s \mathcal{A}$, see [24, Ch. II, §10.B]. The generalization to these contexts of the results presented here will appear elsewhere.

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1. MIDDLE EXTERIOR POWER FORMS

We only consider noetherian schemes. Let X be a scheme with 2 invertible and \mathcal{L} an invertible \mathcal{O}_X -module. An \mathcal{L} -valued *bilinear form* on X is a triple $(\mathcal{E}, b, \mathcal{L})$, where \mathcal{E} is a locally free \mathcal{O}_X -module of finite rank and $b : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{L}$ is an \mathcal{O}_X -module morphism. An \mathcal{L} -valued bilinear form is *symmetric* (resp. *alternating*) if b factors through the canonical epimorphism $\mathcal{E} \otimes \mathcal{E} \rightarrow S^2 \mathcal{E}$ (resp. $\mathcal{E} \otimes \mathcal{E} \rightarrow \bigwedge^2 \mathcal{E}$). As shorthand, a symmetric form will be called $+1$ -symmetric and an alternating form (-1) -symmetric. An \mathcal{L} -valued bilinear form $(\mathcal{E}, b, \mathcal{L})$ is *regular* if the canonical adjoint $\psi_b : \mathcal{E} \rightarrow \mathcal{H}om(\mathcal{E}, \mathcal{L})$ is an \mathcal{O}_X -module isomorphism.

Definition 1.1. Let \mathcal{V} be a locally free \mathcal{O}_X -module of even rank $r = 2s$. Denote by $\lambda^s \mathcal{V} = (\bigwedge^s \mathcal{V}, \wedge, \det \mathcal{V})$ the $\det \mathcal{V}$ -valued *middle exterior power form*,

$$\bigwedge^s \mathcal{V} \otimes \bigwedge^s \mathcal{V} \xrightarrow{\wedge} \bigwedge^r \mathcal{V} = \det \mathcal{V},$$

on X . It is $(-1)^s$ -symmetric, regular, and of rank $n := \binom{r}{s}$.

1.1. Torsorial interpretation. The signed discriminant is defined for line bundle-valued forms of even rank (cf. Parimala–Sridharan [26, §4]). A *discriminant module* consists of a pair (\mathcal{N}, n) , where \mathcal{N} is an invertible \mathcal{O}_X -module and $n : \mathcal{N}^{\otimes 2} \rightarrow \mathcal{O}_X$ is an \mathcal{O}_X -module isomorphism. The isomorphism class of a discriminant module yields a class in $H_{\text{ét}}^1(X, \mu_2)$ (cf. Milne [25, III §4] or Knus [23, III.3]). Applying the determinant functor to the adjoint morphism of a regular line bundle-valued bilinear form $(\mathcal{E}, b, \mathcal{L})$ of rank $n = 2m$ yields an \mathcal{O}_X -module isomorphism

$$\det \mathcal{E} \xrightarrow{\det \psi_b} \det \mathcal{H}om(\mathcal{E}, \mathcal{L}) \xrightarrow{\text{can}} \mathcal{H}om(\det \mathcal{E}, \mathcal{L}^{\otimes n}),$$

giving rise to a discriminant module $\text{disc } b : (\det \mathcal{E} \otimes (\mathcal{L}^\vee)^{\otimes m})^{\otimes 2} \rightarrow \mathcal{O}_X$. As usual, we twist it by $(-1)^m$ to define the *signed discriminant* $d(\mathcal{E}, b, \mathcal{L})$. We denote by $\langle 1 \rangle$ the trivial discriminant module on \mathcal{O}_X .

Perhaps the first fundamental property of a middle exterior power form is that it has trivial signed discriminant.

Lemma 1.2. *Let \mathcal{V} be a locally free \mathcal{O}_X -module of rank $r = 2s$. The middle exterior power form $\lambda^s \mathcal{V}$ has trivial discriminant. Moreover, there is a canonical choice of isomorphism $\zeta_{\mathcal{V}} : d(\lambda^s \mathcal{V}) \rightarrow \langle 1 \rangle$.*

Proof. When \mathcal{V} is free with basis e_1, \dots, e_r a canonical map $\zeta : \det \lambda^s \mathcal{V} \rightarrow \det \mathcal{V}^{\otimes \frac{1}{2} \binom{r}{s}}$ can be given by

$$\bigwedge_{1 \leq i_1 < \dots < i_s \leq r} (e_{i_1} \wedge \dots \wedge e_{i_s}) \mapsto (e_1 \wedge \dots \wedge e_r)^{\otimes \frac{1}{2} \binom{r}{s}}.$$

A standard computation (of the similarity factor of the middle exterior power of the fundamental representation of \mathbf{GL}_r) shows that this map does not depend on the choice of basis. Hence for a general locally free \mathcal{V} , this map patches over a Zariski open covering of X splitting \mathcal{V} . Finally, scaling ζ by $(-1)^m (\det \mathcal{V}^\vee)^{\otimes \frac{1}{2} \binom{r}{s}}$ yields an \mathcal{O}_X -module morphism $\zeta_{\mathcal{V}} : d(\lambda^s \mathcal{V}) \rightarrow \langle 1 \rangle$, which can be checked to be an isometry. \square

A *similarity* between bilinear forms $(\mathcal{E}, b, \mathcal{L})$ and $(\mathcal{E}', b', \mathcal{L}')$ is a pair (φ, λ) consisting of \mathcal{O}_X -module isomorphisms $\varphi : \mathcal{E} \rightarrow \mathcal{E}'$ and $\lambda : \mathcal{L} \rightarrow \mathcal{L}'$ such that either of the following (equivalent) diagrams,

$$(1) \quad \begin{array}{ccc} \mathcal{E} \otimes \mathcal{E} & \xrightarrow{b} & \mathcal{L} \\ \varphi \otimes \varphi \downarrow & & \downarrow \lambda \\ \mathcal{E}' \otimes \mathcal{E}' & \xrightarrow{b'} & \mathcal{L}' \end{array} \quad \begin{array}{ccc} \mathcal{E} & \xrightarrow{\psi_b} & \mathcal{H}om(\mathcal{E}, \mathcal{L}) \\ \varphi \downarrow & & \downarrow \lambda^{-1} \varphi^{\vee \mathcal{L}} \\ \mathcal{E}' & \xrightarrow{\psi_{b'}} & \mathcal{H}om(\mathcal{E}', \mathcal{L}') \end{array}$$

of \mathcal{O}_X -modules commute, where $\lambda^{-1} \varphi^{\vee \mathcal{L}}(\psi) = \lambda^{-1} \circ \psi \circ \varphi$ on sections. Note that the commutativity of the left-hand diagram (1) takes on the familiar formula $b'(\varphi(v), \varphi(w)) = \lambda \circ b(v, w)$ on sections. A similarity transformation (φ, λ) is an *isometry* if $\mathcal{L} = \mathcal{L}'$ and λ is the identity map.

Denote by $\mathbf{Sim}(\mathcal{E}, b, \mathcal{L})$ the presheaf, on the large étale site $X_{\text{ét}}$, of similitudes of a regular \mathcal{L} -valued ± 1 -symmetric form $(\mathcal{E}, b, \mathcal{L})$. In fact, this is a sheaf and is representable by a smooth affine reductive group schemes over X . See Demazure–Gabriel [14, II.1.2.6, III.5.2.3]). Here we consider reductive group schemes whose fibers are not necessarily geometrically integral, in contrast to SGA 3 [1, XIX.2]. When b symmetric, this is the *orthogonal similarity group* $\mathbf{GO}(\mathcal{E}, b, \mathcal{L})$. When b is alternating, this is the *symplectic similarity group* $\mathbf{GSp}(\mathcal{E}, b, \mathcal{L})$. Even though these sheaves of groups are representable by schemes over X , we will still think of them as sheaves of groups on $X_{\text{ét}}$.

Any similarity (φ, λ) between bilinear forms induces an isomorphism of signed discriminants $d(\varphi, \lambda) : d(\mathcal{E}, b, \mathcal{L}) \rightarrow d(\mathcal{E}', b', \mathcal{L}')$. Denote by $\mathbf{Sim}^+(\mathcal{E}, b, \mathcal{L})$ the kernel of the induced homomorphism $\mathbf{Sim}(\mathcal{E}, b, \mathcal{L}) \rightarrow \mathbf{Isom}(d(\mathcal{E}, b, \mathcal{L})) = \mu_2$, i.e. the sheaf of subgroups of similarities (φ, λ) such that $d(\varphi, \lambda) = \text{id}$. If b is alternating, then $\mathbf{Sim}^+(\mathcal{E}, b, \mathcal{L}) = \mathbf{Sim}(\mathcal{E}, b, \mathcal{L})$; the proof in [24, III Prop. 12.3] can be adapted to the étale site. If b is symmetric (and 2 is invertible on X), then $\mathbf{Sim}^+(\mathcal{E}, b, \mathcal{L})$ is called the group of *proper similarities*.

An *oriented bilinear form* $(\mathcal{E}, b, \mathcal{L}, \zeta)$ is a line bundle-valued bilinear form (necessarily of even rank) together with an isomorphism $\zeta : \text{disc}(\mathcal{E}, b, \mathcal{L}) \rightarrow \langle 1 \rangle$ of discriminant modules. In particular, any oriented bilinear form is regular and has trivial signed discriminant. An *oriented similarity* between oriented bilinear forms $(\mathcal{E}, b, \mathcal{L}, \zeta)$ and $(\mathcal{E}', b', \mathcal{L}', \zeta')$ is a triple (φ, λ, ξ) where (φ, λ) is a similarity and $\xi : d(\mathcal{E}, b, \mathcal{L}) \rightarrow d(\mathcal{E}', b', \mathcal{L}')$ is an isomorphism of discriminant modules such that $\zeta' \circ \xi = \zeta$. When X is connected, every regular line bundle-valued symmetric bilinear form with trivial discriminant has two oriented similarity classes.

For any locally free \mathcal{O}_X -module \mathcal{F} , any invertible \mathcal{O}_X -module \mathcal{L} , and any $\epsilon \in \pm 1$, the *hyperbolic form* $H_{\mathcal{L}}^{\epsilon}(\mathcal{F})$ has underlying \mathcal{O}_X -module $\mathcal{F} \oplus \mathcal{H}om(\mathcal{F}, \mathcal{L})$ and bilinear form given by $(v, f) \otimes (w, g) \mapsto f(w) + \epsilon g(v)$ on sections. It is ϵ -symmetric and regular. A hyperbolic form carries a canonical orientation $\zeta_{\mathcal{F}} : d(H_{\mathcal{L}}^{\epsilon}(\mathcal{F})) \rightarrow \langle 1 \rangle$. Every oriented ϵ -symmetric bilinear form of rank $n = 2m$ is locally on $X_{\text{ét}}$ oriented similar to $H_{\mathcal{O}_X}^{\epsilon}(\mathcal{O}_X^m)$, see [3, Thm. 1.14]. For each $n = 2m$, put $\mathbf{Sim}_{m,m} = \mathbf{Sim}(H_{\mathcal{O}_X}(\mathcal{O}_X^m))$ and $\mathbf{Sim}_{m,m}^+ = \mathbf{Sim}^+(H_{\mathcal{O}_X}(\mathcal{O}_X^m))$.

We now put the middle exterior power form into a functorial framework. For $r \geq 1$, denote by $\mathbf{VB}_r(X)$ the groupoid of locally free \mathcal{O}_X -modules of rank r under isomorphism (this is isomorphic to the groupoid of \mathbf{GL}_r -torsors on $X_{\text{ét}}$). For even $n = 2m \geq 1$ and $\epsilon \in \{\pm 1\}$, denote by $\mathbf{BF}_n^{+, \epsilon}(X)$ the groupoid whose objects are oriented line bundle-valued ϵ -symmetric bilinear forms of rank n on X under oriented similarities (this is isomorphic to the groupoid of $\mathbf{Sim}_{m,m}^+$ -torsors over $X_{\text{ét}}$). Denote by \mathbf{VB}_r and $\mathbf{BF}_n^{+, \epsilon}$ the associated stacks over $X_{\text{ét}}$. There are canonical cartesian functors $\det : \mathbf{VB}_r \rightarrow \mathbf{Pic}$ and $\mu : \mathbf{BF}_n^{+, \epsilon} \rightarrow \mathbf{Pic}$ defined by the determinant and value line bundle, respectively, where \mathbf{Pic} is the Picard stack on $X_{\text{ét}}$.

Proposition 1.3. *Let X be a noetherian scheme with 2 invertible, $r = 2s$ even, and $n = \binom{r}{s}$. Then the middle exterior power induces a cartesian functor $\lambda^s : \mathbf{VB}_r \rightarrow \mathbf{BF}_n^{+, (-1)^s}$ making the following diagram*

$$\begin{array}{ccc} \mathbf{VB}_r & \xrightarrow{\lambda^s} & \mathbf{BF}_n^{+, (-1)^s} \\ \det \downarrow & & \downarrow \mu \\ \mathbf{Pic} & \xlongequal{\quad} & \mathbf{Pic} \end{array}$$

of stacks over $X_{\text{ét}}$ commute. In particular for each object \mathcal{V} of $\mathbf{VB}_r(X)$, there's a canonical homomorphism of sheaves of groups $\lambda^s : \mathbf{GL}(\mathcal{V}) \rightarrow \mathbf{Sim}^+(\lambda^s \mathcal{V})$ on $X_{\text{ét}}$, making the following diagram

$$\begin{array}{ccc} \mathbf{GL}(\mathcal{V}) & \xrightarrow{\lambda^s} & \mathbf{Sim}^+(\lambda^s \mathcal{V}) \\ \det \downarrow & & \downarrow \mu \\ \mathbf{G}_m & \xlongequal{\quad} & \mathbf{G}_m \end{array}$$

of sheaves of groups on $X_{\text{ét}}$ commute.

Proof. For each $U \rightarrow X$ in $X_{\text{ét}}$ define $\lambda^s : \mathbf{VB}_r(U) \rightarrow \mathbf{BF}_n^{+, (-1)^s}(U)$ on objects by sending \mathcal{V} to $(\lambda^s \mathcal{V}, \zeta_{\mathcal{V}})$, where $\zeta_{\mathcal{V}}$ is the orientation defined in Lemma 1.2. On morphisms, send an \mathcal{O}_X -isomorphism $\psi : \mathcal{V} \rightarrow \mathcal{V}'$ to the oriented similarity $(\bigwedge^s \psi, \bigwedge^r \psi, \bigwedge^n (\bigwedge^s \psi) \otimes (\bigwedge^r \psi^\vee)^{\otimes \frac{1}{2} \binom{r}{s}})$. The only nontrivial thing to check is that this is indeed an oriented similarity! Then the fact that this gives rise to a cartesian functor of stacks follows from its nature as a tensorial construction. The fact that the diagrams commute follows directly from the definition. \square

1.2. Some linear algebra. Let $\mathcal{V} \xrightarrow{f} \mathcal{N}$ be a morphism of \mathcal{O}_X -modules. For $s \geq 1$, the map $\mathcal{V}^{\otimes s} \rightarrow \mathcal{N} \otimes \mathcal{V}^{\otimes (s-1)}$ defined on sections by

$$v_0 \otimes \cdots \otimes v_{s-1} \mapsto \sum_{j=0}^{s-1} f(v_j) \otimes v_0 \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes v_{s-1}$$

descends to the contraction map $df : \bigwedge^s \mathcal{V} \rightarrow \mathcal{N} \otimes \bigwedge^{s-1} \mathcal{V}$. As usual, define $\bigwedge^0 \mathcal{V} = \mathcal{O}_X$. The composition

$$(2) \quad \bigwedge^s \mathcal{V} \xrightarrow{df} \mathcal{N} \otimes \bigwedge^{s-1} \mathcal{V} \xrightarrow{\text{id}_{\mathcal{N}} \otimes df} \mathcal{N}^{\otimes 2} \otimes \bigwedge^{s-2} \mathcal{V}$$

is zero. Hence for $r \geq 1$ we have an r -truncated Koszul complex

$$0 \rightarrow \bigwedge^r \mathcal{V} \xrightarrow{df} \mathcal{N} \otimes \bigwedge^{r-1} \mathcal{V} \rightarrow \cdots \rightarrow \mathcal{N}^{\otimes (r-2)} \otimes \bigwedge^2 \mathcal{V} \rightarrow \mathcal{N}^{\otimes (r-1)} \otimes \mathcal{V} \rightarrow \mathcal{N}^{\otimes r} \rightarrow 0$$

of \mathcal{O}_X -modules. If \mathcal{V} is locally free of rank r , \mathcal{N} is invertible, and f is an epimorphism, then the r -truncated Koszul complex is exact and denoted by $K(\mathcal{V}, f)$.

Now letting $\mathcal{W} = \ker f \xrightarrow{j} \mathcal{V}$, we have a complex

$$(3) \quad 0 \rightarrow \bigwedge^s \mathcal{W} \xrightarrow{\wedge^s j} \bigwedge^s \mathcal{V} \xrightarrow{df} \mathcal{N} \otimes \bigwedge^{s-1} \mathcal{V}$$

of \mathcal{O}_X -modules, which is exact under the conditions that \mathcal{V} is locally free, \mathcal{N} is invertible, and f is an epimorphism. Such an \mathcal{N} will be called a *line quotient bundle* of \mathcal{V} . The following linear algebra lemma is well-known.

Lemma 1.4 (Pascal's rule for vector bundles). *Let*

$$0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \xrightarrow{f} \mathcal{N} \rightarrow 0,$$

be an exact sequence of locally free \mathcal{O}_X -modules, with \mathcal{N} invertible. Then for any $s \geq 1$ the contraction map induces a short exact sequence,

$$(4) \quad 0 \rightarrow \bigwedge^s \mathcal{W} \rightarrow \bigwedge^s \mathcal{V} \xrightarrow{df} \mathcal{N} \otimes \bigwedge^{s-1} \mathcal{W} \rightarrow 0,$$

of (locally free) \mathcal{O}_X -modules.

Proof. The only thing to check is the image of df in $\mathcal{N} \otimes \bigwedge^{s-1} \mathcal{V}$. By (2) and (3) and our hypotheses, the kernel of $\text{id}_{\mathcal{N}} \otimes df$ is $\mathcal{N} \otimes \bigwedge^{s-1} \mathcal{W}$. By the exactness of the Koszul complex, $\mathcal{N} \otimes \bigwedge^{s-1} \mathcal{W}$ is the image of df . \square

Similarly, if $0 \rightarrow \mathcal{N} \rightarrow \mathcal{V} \xrightarrow{g} \mathcal{W} \rightarrow 0$ is an exact sequence of locally free \mathcal{O}_X -modules, with \mathcal{N} invertible, then there is induced short exact sequence

$$(5) \quad 0 \rightarrow \mathcal{N} \otimes \bigwedge^{s-1} \mathcal{W} \rightarrow \bigwedge^s \mathcal{V} \xrightarrow{\wedge^s g} \bigwedge^s \mathcal{W} \rightarrow 0,$$

by applying Lemma 1.4 to \mathcal{V}^\vee , then dualizing. Such an \mathcal{N} will be call a *line subbundle* of \mathcal{V} .

Remark 1.5. A line subbundle \mathcal{O}_X of \mathcal{V} is also called a *nonvanishing global section*. If X is any noetherian affine scheme of dimension d then Serre's splitting theorem guarantees a nonvanishing global section if \mathcal{V} has rank $> d$. If X is a variety of dimension d over an field and \mathcal{V} has rank $> d$, then \mathcal{V} has a nonvanishing global section as soon as it is generated by global sections (cf. [21, II Exer. 8.2]). Furthermore, if X is projective over a field, then by a theorem of Serre, some twist $\mathcal{V}(n)$ is always generated by global sections (cf. [21, II Thm. 5.17]). Thus for any projective variety X of dimension d over a field and any locally free \mathcal{O}_X -module \mathcal{V} of rank $> d$, there is an exact sequence of the form $0 \rightarrow \mathcal{O}_X(-n) \rightarrow \mathcal{V} \rightarrow \mathcal{W} \rightarrow 0$ for $n \gg 0$, i.e. \mathcal{V} has a line subbundle.

A line bundle-valued bilinear form $(\mathcal{E}, b, \mathcal{L})$ of rank $n = 2m$ is *metabolic* if there exists a locally free \mathcal{O}_X -submodule $\mathcal{F} \rightarrow \mathcal{E}$ with locally free quotient such that the restriction of b to \mathcal{F} is zero. Any choice of such \mathcal{F} is called a *lagrangian*. For example, \mathcal{F} is a lagrangian of the hyperbolic form $H_{\mathcal{L}}^{\epsilon}(\mathcal{F})$. An \mathcal{O}_X -submodule $\mathcal{F} \rightarrow \mathcal{E}$ is a lagrangian if and only if \mathcal{F} is the kernel of $\psi_{b, \mathcal{F}} : \mathcal{E} \rightarrow \mathcal{H}om(\mathcal{F}, \mathcal{L})$ given by the composition of $\psi_b : \mathcal{E} \rightarrow \mathcal{H}om(\mathcal{E}, \mathcal{L})$ and the projection $\mathcal{H}om(\mathcal{E}, \mathcal{L}) \rightarrow \mathcal{H}om(\mathcal{F}, \mathcal{L})$. Thus a lagrangian \mathcal{F} is also equivalent to an isomorphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/\mathcal{F} \longrightarrow 0 \\ & & \downarrow \psi_b|_{\mathcal{F}} & & \downarrow \psi_b & & \downarrow \psi_{b, \mathcal{F}} \\ 0 & \longrightarrow & \mathcal{H}om(\mathcal{H}om(\mathcal{E}/\mathcal{F}, \mathcal{L}), \mathcal{L}) & \longrightarrow & \mathcal{H}om(\mathcal{E}, \mathcal{L}) & \longrightarrow & \mathcal{H}om(\mathcal{V}, \mathcal{L}) \longrightarrow 0 \end{array}$$

of locally free \mathcal{O}_X -modules, where the bottom sequence is the \mathcal{L} -dual of the top sequence. We will often call *lagrangian* a self-dual short exact sequence as above.

For the general notion of metabolic in the context of Grothendieck(-Witt) groups of triangulated and exact categories with duality, see Balmer [5], [6], and [7].

Proposition 1.6. *Let \mathcal{V} be a locally free \mathcal{O}_X -module of rank $r = 2s$ and \mathcal{N} an invertible sheaf.*

a) *For an exact sequence of locally free \mathcal{O}_X -modules,*

$$0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \xrightarrow{f} \mathcal{N} \rightarrow 0,$$

the associated exact sequence (4) is a lagrangian of $\lambda^s \mathcal{V}$.

b) *For an exact sequence of locally free \mathcal{O}_X -modules,*

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{V} \xrightarrow{g} \mathcal{W} \rightarrow 0,$$

the associated exact sequence (5) is a lagrangian of $\lambda^s \mathcal{V}$.

c) *If $\mathcal{V} \cong \mathcal{N} \oplus \mathcal{W}$, then $\lambda^s \mathcal{V}$ is isometric to the hyperbolic form $H_{\det \mathcal{V}}^{(-1)^s}(\wedge^s \mathcal{W})$.*

Proof. This is a linear algebra exercise using the description of lagrangians via self-dual short exact sequences. \square

Corollary 1.7. *If \mathcal{V} has a sub or quotient line bundle then $\lambda^s \mathcal{V}$ is metabolic.*

The statement of Corollary 1.7 can also be seen as a consequence of the Whitney formula (see Fasel [16, Prop. 13.3.2]) for the Euler class in Grothendieck–Witt theory together with explicit formulas (see Fasel–Srinivas [18, Prop. 14]) for its computation. We will draw the connection with the Euler class more fully in §2.

As an easy corollary, we can give a generalization to schemes of a result of Knus [23, V Prop. 5.1.10].

Corollary 1.8. *Let X be any noetherian scheme with the property that any locally free \mathcal{O}_X -module of rank 4 has an invertible sub or quotient (e.g. X has dimension ≤ 3 and is either affine or projective over an infinite field). Then any line bundle-valued symmetric bilinear form of rank 6 with trivial discriminant and trivial Clifford invariant is metabolic.*

Proof. Any line bundle-valued symmetric bilinear form of rank 6 with trivial discriminant is isometric to a reduced pfaffian form (see Bichsel–Knus [11, Thm. 5.2]). Any reduced pfaffian form with trivial Clifford invariant is projectively similar to $\lambda^2 \mathcal{V}$ for some \mathcal{V} of rank 4. Under the hypotheses and Proposition 1.6, any such form is metabolic. But any form projectively similar to a metabolic form is itself metabolic. \square

Remark 1.9. For a general locally free \mathcal{O}_X -module \mathcal{V} , let $p : \mathbb{P}(\mathcal{V}) = \mathbf{Proj} S(\mathcal{V}) \rightarrow X$ be the associated projective bundle. Due to the universal quotient line bundle $p^* \mathcal{V} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$, the form $p^*(\lambda^s \mathcal{V})$ is always metabolic (by Proposition 1.6). If p has a section $\sigma : X \rightarrow \mathbb{P}(\mathcal{V})$ (i.e. $\mathbb{P}(\mathcal{V})(X) \neq \emptyset$) then $\lambda^s \mathcal{V} = \sigma^* p^*(\lambda^s \mathcal{V})$ is metabolic, since the pullback of a metabolic is metabolic. But $\mathbb{P}(\mathcal{V})(X)$ is in bijection with the set of quotient line bundles of \mathcal{V} . Each $\sigma \in \mathbb{P}(\mathcal{V})(X)$ gives rise to the quotient line bundle $\mathcal{V} = \sigma^* p^* \mathcal{V} \rightarrow \sigma^* \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$, providing a specific choice of lagrangian of $\lambda^s \mathcal{V}$. This provides another way of understanding Proposition 1.6.

1.3. Lagrangian grassmannian. The lagrangian grassmannian (or grassmannian of maximal isotropic subspaces) of a bilinear form is a well-studied object. Our perspective is to consider these spaces as moduli functors and as projective homogeneous space fibrations.

A *polarization* on a scheme X is an isomorphism class of \mathcal{O}_X -module \mathcal{F} . An morphism $(f, \varphi) : (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ of polarized schemes is a morphism $f : X \rightarrow Y$ of schemes together with a morphism of \mathcal{O}_Y -modules $\varphi : \mathcal{G} \rightarrow f_*\mathcal{F}$.

Let $(\mathcal{E}, b, \mathcal{L})$ be a regular line bundle-valued ϵ -symmetric bilinear form of even rank $n = 2m$ on X . Denote by $\Lambda\mathbb{G}(\mathcal{E}, b, \mathcal{L})$ its moduli space of lagrangians, called the *lagrangian grassmannian*, and \mathcal{T} the universal lagrangian. The polarized X -scheme $(\Lambda\mathbb{G}(\mathcal{E}, b, \mathcal{L}), \mathcal{T})$ represents the functor

$$u : U \rightarrow X \quad \mapsto \quad \left\{ \text{lagrangians } \mathcal{F} \xrightarrow{\varphi} u^*\mathcal{E} \text{ of } u^*(\mathcal{E}, b, \mathcal{L}) \right\} / \sim$$

where $\mathcal{F} \xrightarrow{\varphi} u^*\mathcal{E} \sim \mathcal{F}' \xrightarrow{\varphi'} u^*\mathcal{E}$ if and only if there exists an \mathcal{O}_U -module isomorphism $\psi : u^*\mathcal{E} \rightarrow u^*\mathcal{E}$ such that $\psi|_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F}'$ is an \mathcal{O}_U -module isomorphism satisfying $\varphi = \varphi' \circ \psi|_{\mathcal{F}}$. Note that if $(\mathcal{E}, b, \mathcal{L})$ is not a split metabolic, then not necessarily every isomorphism $\mathcal{F} \rightarrow \mathcal{F}'$ of lagrangians can be extended to \mathcal{E} .

The stabilizer subgroup $\mathbf{P}_{\mathcal{F}} = \mathbf{P}_{\mathcal{F}}(\mathcal{E}, b, \mathcal{L}) \rightarrow \mathbf{Sim}(\mathcal{E}, b, \mathcal{L})$ of any lagrangian $\mathcal{F} \rightarrow \mathcal{E}$ is a parabolic subgroup. The morphism $\Lambda\mathbb{G}(\mathcal{E}, b, \mathcal{L}) \rightarrow X$ is a projective homogeneous space for the group scheme $\mathbf{Sim}(\mathcal{E}, b, \mathcal{L})$, i.e. a moduli space of parabolic subgroups of $\mathbf{Sim}(\mathcal{E}, b, \mathcal{L})$ of a given type. It has geometrically integral fibers when $\epsilon = -1$ (i.e. in the C_m case). When $\epsilon = 1$ (i.e. the D_m case), it's Stein factorization is of the form $\Lambda\mathbb{G}(\mathcal{E}, b, \mathcal{L}) \xrightarrow{\pi} Z \xrightarrow{f} X$ where π is smooth projective and with geometrically integral fibers and f is the étale double cover of given by the signed discriminant $d(\mathcal{E}, b, \mathcal{L})$. Indeed, by going to geometric points, this is a consequence of the corresponding fact for fields (cf. [15, §85]) or for a more global argument, see Hassett–Várilly-Alvarado–Varilly [22, Prop. 3.3]. When the signed discriminant is trivial, $\Lambda\mathbb{G}(\mathcal{E}, b, \mathcal{L})$ has two connected component, each of which is an $\mathbf{Sim}^+(\mathcal{E}, b, \mathcal{L})$ orbit. In this case, two lagrangians $\mathcal{F} \rightarrow \mathcal{E}$ and $\mathcal{F}' \rightarrow \mathcal{E}$ are points contained in the same connected component if and only if $\mathcal{F}_{\bar{x}} \cap \mathcal{F}'_{\bar{x}} \equiv m \pmod{2}$ for every geometric point \bar{x} of X , see Fulton [19].

Recall that for a locally free \mathcal{O}_X -module \mathcal{V} of rank r , the polarized X -scheme $(\mathbb{P}(\mathcal{V}), \mathcal{O}(1))$, where $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{V})/X}(1)$ is the universal quotient line bundle, represents the moduli functor

$$u : U \rightarrow X \quad \mapsto \quad \left\{ u^*\mathcal{V} \xrightarrow{f} \mathcal{N} : \mathcal{N} \text{ is invertible and } f \text{ is an } \mathcal{O}_U\text{-epimorphism} \right\} / \sim$$

where $u^*\mathcal{V} \xrightarrow{f} \mathcal{N} \sim u^*\mathcal{V} \xrightarrow{f'} \mathcal{N}'$ if and only if there exists an \mathcal{O}_U -module isomorphism $\mu : u^*\mathcal{V} \rightarrow u^*\mathcal{V}$ such that $\mu|_{\mathcal{N}} : \mathcal{N} \rightarrow \mathcal{N}'$ satisfies $f' = \mu|_{\mathcal{N}} \circ f$.

The morphism $\mathbb{P}(\mathcal{V}) \rightarrow X$ is a projective homogeneous space for the sheaf of groups $\mathbf{GL}(\mathcal{V})$ (of type A_{r-1}). The stabilizer subgroup $\mathbf{P}_{\mathcal{W}} = \mathbf{P}_{\mathcal{W}}(\mathcal{V}) \rightarrow \mathbf{GL}(\mathcal{V})$ of any locally free \mathcal{O}_X -module $\mathcal{W} \rightarrow \mathcal{V}$ with invertible quotient is a parabolic subgroup.

Theorem 1.10. *Let X be a scheme with 2 invertible and \mathcal{V} a locally free \mathcal{O}_X -module of even rank $r = 2s$. The associations*

$$\begin{aligned} (0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \xrightarrow{f} \mathcal{N} \rightarrow 0) &\quad \mapsto \quad (0 \rightarrow \bigwedge^s \mathcal{W} \rightarrow \bigwedge^s \mathcal{V} \xrightarrow{\text{df}} \mathcal{N} \otimes \bigwedge^{s-1} \mathcal{W} \rightarrow 0) \\ (0 \rightarrow \mathcal{N} \rightarrow \mathcal{V} \xrightarrow{g} \mathcal{W} \rightarrow 0) &\quad \mapsto \quad (0 \rightarrow \mathcal{N} \otimes \bigwedge^{s-1} \mathcal{W} \rightarrow \bigwedge^s \mathcal{V} \xrightarrow{\wedge^s g} \bigwedge^s \mathcal{W} \rightarrow 0) \end{aligned}$$

induced by Proposition 1.6 are realized by maps on X -points of canonical morphisms of schemes

$$\Phi_{\mathcal{V}} : \mathbb{P}(\mathcal{V}) \rightarrow \Lambda\mathbb{G}(\lambda^s \mathcal{V}), \quad \Phi'_{\mathcal{V}} : \mathbb{P}(\mathcal{V}^{\vee}) \rightarrow \Lambda\mathbb{G}(\lambda^s \mathcal{V}),$$

respectively. Furthermore, in the D_m case, $\Phi_{\mathcal{V}}$ and $\Phi'_{\mathcal{V}}$ map to different connected components.

Proof. There's some easy linear algebra book keeping to make sure the morphism of moduli functor is well-defined. Then Yoneda's lemma shows this to arise from a X -morphism of moduli spaces (or use the classical Plücker embedding, viewing $\Lambda\mathbb{G}(\mathcal{E}, b, \mathcal{L})$ as a closed X -subscheme of the grassmannian $\mathbb{G}(\mathcal{E}, m) \rightarrow X$). The connected component argument is easy using the condition stated above on the mod 2 rank of the intersection of lagrangians. \square

2. THE EULER CLASS IN GROTHENDIECK–WITT THEORY

Let X be a scheme with 2 invertible. Modeled on classical treatments of the Koszul complex (cf. [12, §1.6]), Balmer, Gille, and Nenashev [10], [20] introduce the *Euler class* $e(\mathcal{V}) \in GW^r(X, \det \mathcal{V}^\vee)$ of a locally free \mathcal{O}_X -module \mathcal{V} of rank r . Here we mostly follow Fasel [16] and Fasel–Srinivas [18, §2.4].

2.1. Euler classes and middle exterior power forms. Given a morphism $\mathcal{V} \xrightarrow{f} \mathcal{N}$ of locally free \mathcal{O}_X -modules, with \mathcal{V} of rank r and \mathcal{N} invertible, consider the (twisted) Koszul complex $K(\mathcal{V}, f)$

$$0 \rightarrow (\mathcal{N}^\vee)^{\otimes(r-1)} \otimes \wedge^r \mathcal{V} \rightarrow \cdots \rightarrow \mathcal{N}^\vee \otimes \wedge^2 \mathcal{V} \xrightarrow{df} \mathcal{V} \xrightarrow{f} \mathcal{N} \rightarrow 0$$

which is isomorphic to a twist $\mathcal{N} \otimes K(\mathcal{N}^\vee \otimes \mathcal{V}, \text{ev} \circ (\text{id} \otimes f))$ of a standard Koszul complex associated to the cosection $\mathcal{N}^\vee \otimes \mathcal{V} \rightarrow \mathcal{O}_X$. Note that $K(\mathcal{V}, f) \rightarrow \text{coker}(f)$ is an isomorphism in the derived categories. The perfect pairings $\wedge^j \mathcal{V} \otimes \wedge^{r-j} \mathcal{V} \rightarrow \det \mathcal{V}$ and together with certain sign conventions give rise to a symmetric isomorphism of complexes $\Phi : K(\mathcal{V}, f) \rightarrow K(\mathcal{V}, f)^\sharp$, where $(-)^{\sharp} = \mathcal{H}om(-, (\mathcal{N}^\vee)^{\otimes(r-2)} \otimes \det \mathcal{V})[r]$, see Balmer–Gille [10, §3] or Fasel–Srinivas [18, §2.4].

Consider the associated affine bundle $p : \mathbb{V}(\mathcal{V}) = \mathbf{Spec} S(\mathcal{V}) \rightarrow X$, see [21, II Exer. 5.18], and its zero section $s : X \rightarrow \mathbb{V}(\mathcal{V})$. Then $p^* \mathcal{V}^\vee$ has a canonical “evaluation” \mathcal{O}_X -morphism $f : p^* \mathcal{V}^\vee \rightarrow \mathcal{O}_{\mathbb{V}(\mathcal{V})}$ with cokernel $s_* \mathcal{O}_X$. There's an associated Koszul complex $K(p^* \mathcal{V}^\vee, f)$

$$0 \rightarrow \wedge^r p^* \mathcal{V}^\vee \rightarrow \cdots \rightarrow \wedge^2 p^* \mathcal{V}^\vee \xrightarrow{df} p^* \mathcal{V}^\vee \xrightarrow{f} \mathcal{O}_{\mathbb{V}(\mathcal{V})} \rightarrow 0$$

of locally free $\mathcal{O}_{\mathbb{V}(\mathcal{V})}$ -modules, where $\mathcal{O}_{\mathbb{V}(\mathcal{V})}$ is in degree 0. The canonical sheaf map $s^\sharp : \mathcal{O}_{\mathbb{V}(\mathcal{V})} \rightarrow s_* \mathcal{O}_X$ defines an isomorphism $K(p^* \mathcal{V}^\vee, f) \rightarrow s_* \mathcal{O}_X$ in the derived category. The zero section defines a pullback map

$$s^* : GW^r(\mathbb{V}(\mathcal{V}), p^* \det \mathcal{V}^\vee) \rightarrow GW^r(X, \det \mathcal{V}^\vee)$$

on Grothendieck–Witt groups. The *Euler class* $e(\mathcal{V}) \in GW^r(X, \det \mathcal{V}^\vee)$ is defined to be $s^*(K(p^* \mathcal{V}^\vee, f), \Phi)$.

Proposition 2.1 (Fasel–Srinivas [18, Prop. 14, 21]). *Let X be a scheme with 2 invertible and \mathcal{V} a locally free \mathcal{O}_X -module of rank r .*

a) *(Explicit formula for Euler classes) In $GW^r(X, \det \mathcal{V}^\vee)$, we have the equalities*

$$e(\mathcal{V}) = \begin{cases} H_{\det \mathcal{V}^\vee} \left(\sum_{j=0}^s (-1)^j [\wedge^j \mathcal{V}^\vee] \right) & \text{if } r = 2s + 1 \\ \langle (-1)^{s(s-1)/2} \rangle \otimes \lambda^s \mathcal{V}^\vee + H_{\det \mathcal{V}^\vee} \left(\sum_{j=0}^{s-1} (-1)^j [\wedge^j \mathcal{V}^\vee] \right) & \text{if } r = 2s \end{cases}$$

where $\lambda^s \mathcal{V}^\vee$ is considered as a complex concentrated in degree s .

b) *(Whitney sum formula for Euler classes) If $0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \rightarrow \mathcal{N} \rightarrow 0$ is an exact sequence of locally free \mathcal{O}_X -modules, then $e(\mathcal{V}) = e(\mathcal{W}) \cdot e(\mathcal{N})$ under the multiplication in Grothendieck–Witt groups induced by $\det \mathcal{W}^\vee \otimes \det \mathcal{N}^\vee \rightarrow \det \mathcal{V}^\vee$.*

Remark 2.2. Tensoring with $\det \mathcal{V}^\vee$ — actually multiplication in the Grothendieck–Witt group by the bilinear form $(\det \mathcal{V}^\vee, \otimes, (\det \mathcal{V}^\vee)^{\otimes 2})$ of rank 1 — yields a canonical isomorphism of groups $GW^r(X, \det \mathcal{V}) \rightarrow GW^r(X, \det \mathcal{V}^\vee)$ via the evaluation morphism $\det \mathcal{V} \otimes (\det \mathcal{V}^\vee)^{\otimes 2} \rightarrow \det \mathcal{V}^\vee$. Under this isomorphism the class of $\lambda^s \mathcal{V}$ is mapped to the class of $\lambda^s \mathcal{V}^\vee$. Also note that $\lambda^s \mathcal{V}$ is metabolic if and only if $\langle \pm 1 \rangle \otimes \lambda^s \mathcal{V}$ is metabolic.

The following corollary can be viewed as a generalization of Calmès–Hornbostel [13, Rem. 7.5].

Corollary 2.3. *If \mathcal{V} has a sub or quotient line bundle then $e(\mathcal{V})$ is metabolic.*

Proof. Combine Proposition 2.1a) and Corollary 1.7. Another proof employs the Whitney sum formula for Euler classes, noting that the product of metabolic classes is metabolic. \square

2.2. Euler classes and quotient line bundles. For any invertible \mathcal{O}_X -module \mathcal{N} , Karoubi periodicity gives rise to the exact sequence

$$(6) \quad GW^{r-1}(X, \mathcal{N}) \xrightarrow{f} K_0(X) \xrightarrow{H_{\mathcal{N}}} GW^r(X, \mathcal{N}) \rightarrow W^r(X, \mathcal{N}) \rightarrow 0$$

where f is the forgetful map, see Walter [28, Thm. 2.6]. If a locally free \mathcal{O}_X -module \mathcal{V} of even rank $r = 2s$ has a sub or quotient line bundle, then by Corollary 2.3, the Euler class $e(\mathcal{V})$ is in the image of the hyperbolic map:

$$e(\mathcal{V}) = H_{\det \mathcal{V}^\vee} \left([\wedge^s \mathcal{V}^\vee] + \sum_{j=0}^{s-1} (-1)^j [\wedge^j \mathcal{V}^\vee] \right).$$

We can give a potentially useful representation of this class. For a locally free \mathcal{O}_X -module \mathcal{W} , denote by $\wedge \mathcal{W}$ the class

$$\sum_{j=0}^{\text{rank } \mathcal{W}} (-1)^j [\wedge^j \mathcal{W}^\vee]$$

in $K_0(X)$. Recall that $\wedge \mathcal{W} = s^* s_* \mathcal{O}_X$, where $s : X \rightarrow \mathbb{V}(\mathcal{W})$ is the zero section and s_* and s^* are the associated pushforward and pullback maps on K_0 .

Proposition 2.4. *Let \mathcal{V} be a locally free \mathcal{O}_X -module of rank r and suppose that $0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \rightarrow \mathcal{L} \rightarrow 0$ is a short exact sequence of locally free \mathcal{O}_X -modules, with \mathcal{L} invertible. Then we have*

$$e(\mathcal{V}) = H_{\det \mathcal{V}^\vee} (\wedge \mathcal{W})$$

in $GW^r(X, \det \mathcal{V}^\vee)$.

Proof. The proof is a straightforward calculation using the explicit formula for Euler classes (Proposition 2.1a), Lemma 1.4, and the fact that, for all $j \geq 1$,

$$[\mathcal{L}^\vee \otimes \wedge^{j-1} \mathcal{W}^\vee] - [\wedge^{r-j} \mathcal{W}^\vee]$$

is in the kernel of the hyperbolic map $H_{\det \mathcal{V}^\vee} : K_0(X) \rightarrow GW^r(X, \det \mathcal{V}^\vee)$. A proof could probably also be given using the Whitney sum formula for Euler classes. \square

A similar formula (for the case of \mathcal{V} of odd rank) appears in Fasel [17, Thm. 10.1]. As a consequence of Proposition 2.4, we can give a direct proof of a fact used in the proof of Fasel–Srinivas [18, Prop. 22].

Corollary 2.5. *Under the hypotheses of Proposition 2.4, if \mathcal{O}_X is in the kernel of the hyperbolic map $H_{\mathcal{L}^\vee} : K_0(X) \rightarrow GW^1(X, \mathcal{L}^\vee)$, then $e(\mathcal{V}) = 0$.*

Proof. Let $p : \mathbb{V}(\mathcal{W}) \rightarrow X$ be the projection and s the zero section. Then by Proposition 2.4 we have

$$e(\mathcal{V}) = H_{\det \mathcal{V}^\vee}(s^* s_* \mathcal{O}_X) = s^* H_{p^* \det \mathcal{V}^\vee}(s_* \mathcal{O}_X) = s^* s_* H_{\mathcal{L}^\vee}(\mathcal{O}_X)$$

using the pushforward in Grothendieck–Witt groups, which can be applied since we have the isomorphism

$$i^! p^* \det \mathcal{V}^\vee = i^* p^* \det \mathcal{V}^\vee \otimes \omega_s \cong \det \mathcal{V}^\vee \otimes \det \mathcal{W} \cong \mathcal{L}^\vee$$

of invertible \mathcal{O}_X -modules. \square

3. RANK FOUR VECTOR BUNDLES

In this section, we apply the above general results to the specific case of locally free \mathcal{O}_X -modules of rank four. In this special situation, we are helped by the accidental correspondence $A_3 = D_3$.

3.1. Middle exterior forms. Let X be a scheme with 2 invertible, and recall that by Proposition 1.3 the middle exterior power functor gives rise to a canonical homomorphism $\lambda^2 : \mathbf{GL}(\mathcal{V}) \rightarrow \mathbf{GSO}(\lambda^2 \mathcal{V})$. When \mathcal{V} has rank four, this homomorphism is an isogeny.

Proposition 3.1. *Let \mathcal{V} be a locally free \mathcal{O}_X -module of rank 4, then there's a short exact sequence*

$$1 \rightarrow \mu_2 \rightarrow \mathbf{GL}(\mathcal{V}) \xrightarrow{\lambda^2} \mathbf{GSO}(\lambda^2 \mathcal{V}) \rightarrow 1$$

of sheaves of groups on $X_{\text{ét}}$.

Proof. The only thing to check is that λ^2 is an epimorphism on $X_{\text{ét}}$, which is a classical fact. See Knus [23, V.5.6] for instance. \square

Remark 3.2. The coboundary map $H_{\text{ét}}^1(X, \mathbf{GSO}(\lambda^2 \mathcal{V})) \rightarrow H_{\text{ét}}^2(X, \mu_2)$ associates, to any oriented regular line bundle-valued symmetric bilinear form $(\mathcal{E}, b, \mathcal{L}, \zeta)$ of rank 6, an invariant $gc(\mathcal{E}, b, \mathcal{L}, \zeta) \in H_{\text{ét}}^2(X, \mu_2)$, and is an example of the oriented similarity Clifford invariant studied in [4].

The D_3 version of Theorem 1.10 is more precise, yielding a stronger version of Corollary 1.7 as follows.

Theorem 3.3. *Let X be a scheme with 2 invertible and \mathcal{V} a locally free \mathcal{O}_X -module of rank 4. Then*

$$\Phi_{\mathcal{V}} \sqcup \Phi'_{\mathcal{V}} : \mathbb{P}(\mathcal{V}) \sqcup \mathbb{P}(\mathcal{V}^\vee) \rightarrow \Lambda \mathbb{G}(\lambda^2 \mathcal{V})$$

is an isomorphism of X -schemes.

Proof. We can reduce to the classical case over fields by going to geometric points. We prefer to argue directly. Let $\mathbf{P} \hookrightarrow \mathbf{GL}_4$ be the parabolic subgroup given by the stabilizer of $\mathcal{O}_X^3 \subset \mathcal{O}_X^4$ and let $\mathbf{Q} \hookrightarrow \mathbf{GSO}_{3,3}$ be the parabolic subgroup corresponding to the associated choice of oriented lagrangian $\bigwedge^2 \mathcal{O}_X^3$ of $\lambda^2 \mathcal{O}_X^4 \cong H_{\mathcal{O}_X}(\mathcal{O}_X^3)$. Then upon restricting the morphism λ^2 , we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbf{P} & \xrightarrow{\lambda^2} & \mathbf{Q} \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbf{GL}_4 & \xrightarrow{\lambda^2} & \mathbf{GSO}_{3,3} \longrightarrow 1 \end{array}$$

of groups schemes on X . Similarly, we have a epimorphism of (right) torsors

$$\mathbf{Isom}(\mathcal{O}_X^4, \mathcal{V}) \xrightarrow{\lambda^2} \mathbf{Sim}^+(\lambda^2 \mathcal{O}_X^4, \lambda^2 \mathcal{V})$$

equivariant for the corresponding homomorphism of group schemes. Then in this situation, we have an induced commutative diagram of X -scheme isomorphisms

$$\begin{array}{ccc} \mathbf{Isom}(\mathcal{O}_X^4, \mathcal{V})/\mathbf{P} & \xrightarrow{\sim} & \mathbf{Sim}^+(\lambda^2 \mathcal{O}_X^4, \lambda^2 \mathcal{V})/\mathbf{Q} \\ \wr \downarrow & & \downarrow \wr \\ \mathbb{P}(\mathcal{V}) & \xrightarrow{\Phi_{\mathcal{V}}} & \Lambda\mathbb{G}(\lambda^2 \mathcal{V})^\circ \end{array}$$

where $\Lambda\mathbb{G}(\lambda^2 \mathcal{V})^\circ$ is the connected component containing the image of $\Phi_{\mathcal{V}}$ and the vertical isomorphism are a consequence of SGA 3 [1, XXVI.3 Lemma 3.2], since these projective homogeneous schemes are moduli spaces of parabolic subgroups of the associated groups.

Note that if $\mathbf{GL}(\mathcal{V})$ actually had a parabolic subgroup $\mathbf{P}_{\mathcal{V}}$ of the same type as \mathbf{P} (i.e. there exists an exact sequence $0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \rightarrow \mathcal{L} \rightarrow 0$ with \mathcal{L} invertible), then the above argument can be summarized with the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbf{P}_{\mathcal{V}} & \xrightarrow{\lambda^2} & \mathbf{P} \wedge^2 \mathcal{V} \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbf{GL}(\mathcal{V}) & \xrightarrow{\lambda^2} & \mathbf{GSO}(\lambda^2 \mathcal{V}) \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathbb{P}(\mathcal{V}) & \xrightarrow{\Phi_{\mathcal{V}}} & \Lambda\mathbb{G}(\lambda^2 \mathcal{V})^\circ \end{array}$$

of X -scheme, where the top two rows are short exact sequences of group schemes.

So far, we've shown that $\Phi_{\mathcal{V}}$ is an isomorphism onto a connected component. The same argument can be used for $\Phi'_{\mathcal{V}}$ (using the dual parabolic subgroup of \mathbf{P}). Since $\Lambda\mathbb{G}(\lambda^2 \mathcal{V}) \rightarrow X$ has two connected X -components (since the discriminant is trivial) and $\Phi_{\mathcal{V}}$ and $\Phi'_{\mathcal{V}}$ map to different components, we are done. \square

Corollary 3.4. *Let X be a scheme with 2 invertible and \mathcal{V} a locally free \mathcal{O}_X -module of rank 4. Then \mathcal{V} has a sub or quotient line bundle if and only if $\lambda^2 \mathcal{V}$ is a metabolic form.*

Definition 3.5. Let \mathcal{V} be a locally free \mathcal{O}_X -module of rank r . Define $\lambda(\mathcal{V}) \in GW^r(X, \det \mathcal{V})$ to be the class of $\lambda^s \mathcal{V}$ if $r = 2s$ is even and 0 if r is odd.

Of course, if $r = 2s$ is even and $\lambda^s \mathcal{V}$ is metabolic, then the Witt class $\lambda(\mathcal{V}) \in W^r(X, \det \mathcal{V})$ vanishes (see Remark 2.2). In general, the converse — that if $\lambda^s \mathcal{V}$ is stably metabolic then it is metabolic — may not hold. However, under certain hypotheses on X , the converse does indeed hold.

Definition 3.6. Let $\mathbf{C} = (\mathbf{C}, \sharp, \varpi)$ be an exact category with duality (cf. Balmer [8, §1.1.2]). We say that the *Witt cancellation property* is satisfied over \mathbf{C} if given symmetric objects $(\mathcal{E}_1, b_1), (\mathcal{E}_2, b_2), (\mathcal{E}_3, b_3), (\mathcal{E}_4, b_4)$ in \mathbf{C} such that $(\mathcal{E}_1, b_1) \perp (\mathcal{E}_2, b_2)$ is isometric to $(\mathcal{E}_3, b_3) \perp (\mathcal{E}_4, b_4)$ and (\mathcal{E}_2, b_2) is isometric to (\mathcal{E}_4, b_4) then we have that (\mathcal{E}_1, b_1) is isometric to (\mathcal{E}_3, b_3) .

We say that the *Krull–Schmidt property* is satisfied over an exact category \mathbf{C} if every object has a unique (up to permutation) coproduct decomposition into indecomposable objects. Additionally, we say that the *strong Krull–Schmidt property* is

satisfied if for every object \mathcal{E} of \mathbf{C} , the ring $\text{End}_{\mathbf{C}}(\mathcal{E})$ is complete with respect to the $\text{rad}(\text{End}_{\mathbf{C}}(\mathcal{E}))$ -adic topology. If X is proper over a complete local ring (in particular a field), then the strong Krull–Schmidt property is satisfied over the exact category $\text{VB}(X)$ of locally free \mathcal{O}_X -modules, see Atiyah [2, Thm. 3].

Theorem 3.7 (Knus [23, II Thm. 6.6.1]). *Let \mathbf{C} be any exact category with duality over which the strong Krull–Schmidt property is satisfied. Then \mathbf{C} satisfies the Witt cancellation property.*

Corollary 3.8. *Let X be a scheme with 2 invertible such that $\text{VB}(X)$ satisfies the strong Krull–Schmidt property (e.g. X is proper over a complete local ring with residue field of characteristic not 2). Let \mathcal{V} be a locally free \mathcal{O}_X -module of rank 4. Then \mathcal{V} has a sub or quotient line bundle if and only if $\lambda(\mathcal{V}) = 0$ in $W(X, \det \mathcal{V})$.*

Proof. This is direct consequence of Corollary 3.4 and the Theorem 3.7. \square

Remark 3.9. Note that under the hypotheses of Corollary 3.8, $\lambda(\mathcal{V})$ vanishes in $W(X, \det \mathcal{V})$ if and only if $e(\mathcal{V})$ vanishes in $W(X, \det \mathcal{V}^\vee)$. Indeed, Proposition 2.1 implies that $e(\mathcal{V})$ vanishes if $\lambda(\mathcal{V})$ does, the converse follows from the Witt cancellation property. Thus \mathcal{V} has a sub or quotient line bundle if and only if $e(\mathcal{V}) = 0$ in $W(X, \det \mathcal{V}^\vee)$. In particular, this is the case if $e(\mathcal{V}) = 0$ in $GW(X, \det \mathcal{V}^\vee)$.

3.2. A question on the vanishing of Euler classes. The following problem arises naturally from this work. Let \mathcal{V} be a locally free \mathcal{O}_X -module of rank 4 and suppose there's an exact sequence $0 \rightarrow \mathcal{W} \rightarrow \mathcal{V} \rightarrow \mathcal{L} \rightarrow 0$. Then by Propositions 2.1 and 2.4, we have

$$e(\mathcal{V}) = H_{\det \mathcal{V}^\vee}(\bigwedge^2 \mathcal{W}^\vee + \mathcal{O}_X - \mathcal{V}^\vee) = H_{\det \mathcal{V}^\vee}(\mathcal{O}_X - \mathcal{W}^\vee + \bigwedge^2 \mathcal{W}^\vee - \det \mathcal{W}^\vee).$$

Problem 3.10. *Give a complete characterization of the vanishing of the Euler class $e(\mathcal{V}) \in GW^4(X, \det \mathcal{V}^\vee)$ in terms of additional properties of \mathcal{W} and \mathcal{L} .*

We already know $\mathcal{L} = \mathcal{O}_X$ is a sufficient condition for the vanishing of the Euler class. By exact sequence (6), $e(\mathcal{V})$ vanishes if and only if $\bigwedge \mathcal{W}$ is the underlying complex of an element of $GW^3(X, \det \mathcal{V}^\vee)$.

By Proposition 2.4, we know that $\bigwedge \mathcal{W}$ is the underlying complex of the Euler class $e(\mathcal{W}) \in GW^3(X, \det \mathcal{W}^\vee)$. Since $\det \mathcal{V} \cong \det \mathcal{W} \otimes \mathcal{L}$, this gives another way of seeing that if $\mathcal{L} = \mathcal{O}_X$ then $e(\mathcal{V})$ is trivial (since then there's an isomorphism of groups $GW^i(X, \det \mathcal{W}^\vee) \cong GW^i(X, \det \mathcal{V}^\vee)$ preserving underlying complexes).

More generally, consider the case $\mathcal{L} \cong \mathcal{N}^{\otimes 2}$ (and we fix such an isomorphism). Then tensoring by \mathcal{N}^\vee yields an isomorphism (cf. Balmer–Calmès [9]) of groups $GW^i(X, \det \mathcal{W}^\vee) \rightarrow GW^i(X, \det \mathcal{V}^\vee)$. Thus in this case, $e(\mathcal{V})$ vanishes if and only if $\bigwedge \mathcal{W}$ is the underlying complex of an element in $GW^3(X, \det \mathcal{V}^\vee)$ if and only if $\mathcal{N} \otimes \bigwedge \mathcal{W}$ is the underlying complex of an element in $GW^3(X, \det \mathcal{W}^\vee)$. For examples, this happens if $\mathcal{N} \in {}_2\text{Pic}(X)$ and also if $\mathcal{N} \cong \det \mathcal{W}^\vee \cong \det \mathcal{V}$. What is a necessary condition on \mathcal{N} ensuring the vanishing of $e(\mathcal{V})$ in the case?

REFERENCES

1. *Schémas en groupes. III: Structure des schémas en groupes réductifs*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin, 1962/1964.
2. M. Atiyah, *On the Krull-Schmidt theorem with application to sheaves*, Bull. Soc. Math. France **84** (1956), 307–317.
3. Asher Auel, *Cohomological invariants of line bundle-valued symmetric bilinear forms*, Ph.D. thesis, University of Pennsylvania, Philadelphia, 2009.

4. ———, *Clifford invariants of line bundle-valued quadratic forms*, MPIM preprint series 2011 (33), May 2011.
5. Paul Balmer, *Derived Witt groups of a scheme*, J. Pure Appl. Algebra **141** (1999), 101–129.
6. ———, *Triangular Witt groups. I. The 12-term exact sequence*, K-Theory **19** (2000), 311–363.
7. ———, *Triangular Witt groups. II. From usual to derived*, Math. Z. **236** (2001), 351–382.
8. Paul Balmer, *Witt groups*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 539–576.
9. Paul Balmer and Baptiste Calmès, *Bases of total Witt groups and lax-similitude*, preprint arXiv:1104.5051v1, April 2011.
10. Paul Balmer and Stefan Gille, *Koszul complexes and symmetric forms over the punctured affine space*, Proc. London Math. Soc. (3) **91** (2005), no. 2, 273–299.
11. W. Bichsel and M.-A. Knus, *Quadratic forms with values in line bundles*, Contemp. Math. **155** (1994), 293–306.
12. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.
13. Baptiste Calmès and Jens Hornbostel, *Push-forwards for Witt groups of schemes*, preprint, arXiv:0806.0571v1 [math.AG], 2008.
14. Michel Demazure and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Avec un appendice, Corps de classes local, par Michiel Hazewinkel*, Masson & Cie, Éditeur, Paris; North-Holland Publishing Company, Amsterdam, 1970.
15. Richard Elman, Nikita Karpenko, and Alexander Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008.
16. Jean Fasel, *Groupes de Chow-Witt*, Mém. Soc. Math. Fr. (N.S.) (2008), no. 113, viii+197.
17. Jean Fasel, *The projective bundle theorem for I^1 -cohomology*, preprint, 2009.
18. Jean Fasel and V. Srinivas, *Chow-Witt groups and Grothendieck-Witt groups of regular schemes*, Adv. Math. **221** (2009), 302–329.
19. William Fulton, *Schubert varieties in flag bundles for the classical groups*, Proceedings of Conference in Honor of Hirzebruch's 65th Birthday, Bar Ilan, 1993, vol. 9, Israel Mathematical Conference Proceedings, 1995.
20. Stefan Gille and Alexander Nenashev, *Pairings in triangular Witt theory*, J. Algebra **261** (2003), no. 2, 292–309.
21. Robin Hartshorne, *Algebraic geometry*, Graduate Texts Math., vol. 52, Springer-Verlag, New York, 1977.
22. Brendan Hassett, Anthony Várilly-Alvarado, and Patrick Varilly, *Transcendental obstructions to weak approximation on general K3 surfaces*, Adv. in Math., to appear, 2011.
23. Max-Albert Knus, *Quadratic and hermitian forms over rings*, Springer-Verlag, Berlin, 1991.
24. Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, Colloquium Publications, vol. 44, AMS, 1998.
25. J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, no. 33, Princeton University Press, Princeton, N.J., 1980.
26. R. Parimala and R. Sridharan, *Reduced norms and pfaffians via Brauer-Severi schemes*, Contemp. Math. **155** (1994), 351–363.
27. Charles Walter, *Grothendieck-Witt groups of projective bundles*, preprint, K-theory preprint archive, 2003.
28. ———, *Grothendieck-Witt groups of triangulated categories*, preprint, K-theory preprint archive, 2003.

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